16. Cayley's theorem

Dummit & Foote §4.2

Theorem 16.0.18. *Every finite group of order n is isomorphic to a subgroup of S_n.*

We first prove a lemma that puts group actions in a different context. Let A be a finite set. Let Σ_A be the set of bijective functions $A \longrightarrow A$. Then, Σ_A is a group. In fact, if we let $s_1, \ldots, s_n$ be the elements of A, we can identify bijective functions $A \longrightarrow A$ with permutations of $\{1, \ldots, n\}$ and we see that $\Sigma_A \cong S_n$.

Lemma 16.0.19. *To give an action of a group G on a set A is equivalent to giving a homomorphism $G \longrightarrow \Sigma_A$. The kernel of this homomorphism is $\cap_{a \in A} Stab(a)$.* Dummit & Foote §4.1

Proof. An element g define a function $\phi_g : A \longrightarrow A$ by $\phi_g(a) = ga$. We have ϕ_e being the identity function. Note that $\phi_h \phi_g(a) = \phi_h(ga) = hga = \phi_{hg}(a)$ for every a and hence $\phi_h \phi_g = \phi_{hg}$. In particular, $\phi_g \phi_{g^{-1}} = \phi_{g^{-1}} \phi_g = Id$. This shows that every ϕ_g is a bijection and the map

$$\Psi : G \longrightarrow \Sigma_A, \qquad g \overset{\Psi}{\longmapsto} \phi_g,$$

is a homomorphism. (Conversely, given such a homomorphism Ψ, define a group action by $g * a := \Psi(g)(a)$.)

The kernel of this homomorphism is the elements g such that ϕ_g is the identity, i.e., $\phi_g(a) = a$ for all $a \in A$. That is, $g \in Stab(a)$ for every $a \in A$. The set of such elements g is just $\cap_{a \in A} \mathrm{Stab}(a)$. □

Proof. (*of Theorem*) Consider the action of G on itself by multiplication (Example 15.0.14), $(g, g') \longmapsto gg'$. Recall that all stabilizers are trivial. Thus this action gives an injective homomorphism

$$G \longrightarrow \Sigma_G \cong S_n,$$

where $n = |G|$. □

16.1. **Applications to construction of normal subgroups.** Let G be a group and H a subgroup of finite index n. Consider the action of G on the set of cosets G/H of H and the resulting homomorphism

$$\Psi : G \longrightarrow \Sigma_{G/H} \cong \Sigma_n,$$

where $n = [G : H]$. The kernel K of Ψ is

$$\cap_{a \in G/H} \mathrm{Stab}(a) = \cap_{g \in G} \mathrm{Stab}(gH) = \cap_{g \in G} gHg^{-1}.$$

Being a kernel of a homomorphism, K is normal in G and is contained in H. Furthermore, since the resulting homomorphism $G/K \longrightarrow S_n$ is injective we get that $|G/K| = [G : K]$ divides $[G : H]! = |S_n|$. In particular, we conclude that every subgroup H of G contains a subgroup K which is normal in G and of index at most $[G : H]!$. Thus, for example, a simple infinite group has no subgroups of finite index.

In fact, the formula $K = \cap_{g \in G} gHg^{-1}$ shows that K is the maximal subgroup of H which is normal in G. Indeed, if $K' \lhd G, K' < H$ then $K = gKg^{-1} \subset gHg^{-1}$ and we see that $K' \subseteq K$.

Scanned by TapScanner