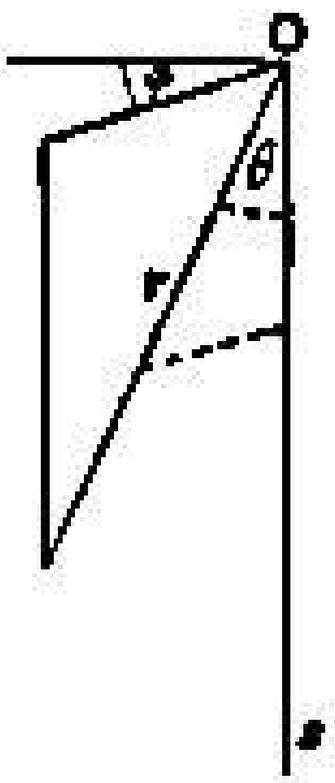
6·41. Motion of a uniform heavy rod which is free to turn about an end which is fixed. Let $2a$ be the length of the rod, O the fixed end and Oz vertical, M the whole mass and m the mass of a small element at a distance r from the fixed end; θ the inclination to the vertical and ϕ the angle which the vertical plane through the rod makes with a fixed vertical plane.



Let us find the condition that the rod may rotate uniformly so that ϕ and θ may have constant values ω and α.

The moment of momentum about Oz is constant, since the weight and the reaction at O have no moment about Oz. Hence

$$\Sigma(mr^2\sin^2\theta.\dot{\phi}) = \text{const.},$$

or
$$\dot{\phi}\sin^2\theta = \omega\sin^2\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

for Σmr^2 is a constant $\frac{1}{3}Ma^2$.

Again, there is an energy equation

$$\tfrac{1}{2}\Sigma\{mr^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)\} - Mga\cos\theta = \text{const.},$$

$$\text{or}\quad \tfrac{1}{3}Ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta) - Mga\cos\theta = \tfrac{1}{3}Ma^2\omega^2\sin^2\alpha - Mga\cos\alpha \quad \ldots\ldots(2).$$

If we eliminate $\dot{\phi}$ between (1) and (2), we get

$$\tfrac{1}{3}a\left(\dot{\theta}^2+\frac{\omega^2\sin^4\alpha}{\sin^2\theta}\right) - g\cos\theta = \tfrac{1}{3}a\omega^2\sin^2\alpha - g\cos\alpha \quad (3).$$

Differentiating and dividing by $\dot{\theta}$, we find that

$$4a(\ddot{\theta} - \omega^2\sin^4\alpha\cot\theta\operatorname{cosec}^2\theta) + 3g\sin\theta = 0 \quad \ldots(4).$$

For steady motion this must be satisfied by $\theta = \alpha$, so that the required condition is

$$4a\omega^2\cos\alpha = 3g \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

For small oscillations about this state of steady motion, we put $\theta = \alpha + x$, where x is small, in (4), and retain only the first power of x. This substitution is most simply effected by observing that equation (4) is of the form

$$\ddot{\theta} + f(\theta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

which is equivalent to $\ddot{x} + f(\alpha + x) = 0,$

or $\ddot{x} + f(\alpha) + xf'(\alpha) = 0,$

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and $f(x)$ vanishes because of (5), so that (6) reduces to

$$\ddot{\chi} + \chi f'(x) = 0;$$

or, in this case, $\ddot{\chi} + \omega^2(1 + 3\cos^2\alpha)\chi = 0,$

which represents an oscillatory motion with period

$$2\pi/\omega\sqrt{(1 + 3\cos^2\alpha)}.$$

6·42. Application of the Principle of Virtual Work. Since in any configuration of a moving system the effective forces are the exact equivalent of the impressed forces, it follows that if we consider any possible virtual displacement of the system from this configuration the virtual work of the effective forces must be equal to that of the impressed forces; provided that we include in this equation of virtual work every force whether internal or external which does work in the displacement.

Consider for example a single rigid body. Suppose that it possesses a rate of change of linear momentum F in an assigned direction and that δs is the virtual displacement of the centre of gravity in that direction, then a term $F\delta s$ enters the equation of virtual work. Similarly, if it possesses a rate of change of moment of momentum H about an assigned axis through its centre of gravity and $\delta\phi$ is the virtual rotation about that axis, the corresponding term in the equation of virtual work is $H\delta\phi$.

6·43. Example. *A uniform elastic ring of mass m, length $2\pi c$ and modulus of elasticity mg is placed horizontally on a smooth paraboloid of revolution of latus rectum $4a$ with axis vertical and vertex uppermost. Find the equation of motion, the position of equilibrium and the time of a small oscillation about it.*

Let $y^2 = 4ax$ be the equation of the meridian section of the paraboloid, T the tension of the ring when its radius is y. If s denotes an arc of the meridian curve, any point of the ring has acceleration $\ddot{s}$ and T/ρ, and if we consider a virtual displacement of the ring downwards we have an equation of virtual work

$$m\ddot{s}\delta s = mg\delta x - T\delta(2\pi y),$$

or
$$\ddot{s} = g\frac{dx}{ds} - 2\pi g\frac{y - c}{c}\frac{dy}{ds} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This equation might have been found by considering an element of the string, or by writing down the equation of energy and differentiating it.

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Expressing $\dot{y}$ and x in terms of y, we get

$$\dot{y}^2(y^2+4a^2)+y\dot{y}^2=2agy-8\pi ga^2(y-c)/c \quad \ldots\ldots\ldots\ldots(2).$$

Hence the equilibrium position is given by

$$y_0=4\pi ac/(4\pi a-c) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

For small oscillations we put $y=y_0+\eta$, where η is small, and retain only the first power of η. Thus

$$\ddot{\eta}(y_0^2+4a^2)=2a\eta g\left(1-\frac{4\pi a}{c}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Since (3) requires $4\pi a-c$ to be positive, (4) represents an oscillatory motion of period $2\pi\left[\dfrac{2ca\{4\pi^2c^2+(4\pi a-c)^2\}}{g(4\pi a-c)^3}\right]^{\frac{1}{2}}$.

6·44. Examples. 1. A rod of length $2a$ moves under gravity about one end which is smoothly jointed to a fixed point, and has its other end joined by an elastic string, of natural length a and modulus of elasticity equal to the weight of the rod, to a second point at a distance a vertically below the first. Prove that the angular velocity with which the rod must rotate about the vertical in order that the string may be stretched to twice its natural length is $\sqrt{(6g/a)}$, and that the time of a small oscillation about the steady motion is $16\pi\sqrt{(2a/957g)}$. [L. 1906]

2. Four equal uniform rods AB, BC, CD, DA, each of length $2a$, are smoothly hinged so as to form a rhombus, and are suspended by the hinge at A. The hinges B, D are connected by an elastic string whose modulus is twice the weight of a rod, and whose natural length is $a\sqrt{2}$. The whole system revolves round the vertical diagonal AC. Determine the angular velocity ω when, in a state of steady motion, the rods form a square; and shew that if C be slightly disturbed vertically, the period of a small oscillation is $2\pi/\omega$. [I. 1898]

3. An elliptic cylinder of semi-axes a and b and radius of gyration k about the line of centres rolls on a horizontal plane.

Prove that $(r^2+k^2)\omega^2+2gp=\text{constant}$,

where r is the distance of the point of contact from the line of centres, ω is the angular velocity, and p the height of the line of centres above the plane. Shew also that ω and p are related to ϕ, the eccentric angle of the point of contact, by the equations

$$\omega^2=-\frac{ab}{a^2\sin^2\phi+b^2\cos^2\phi}\dot{\phi},$$

$$p^2=\frac{a^2b^2}{a^2\sin^2\phi+b^2\cos^2\phi}.$$

Explain briefly why in this case the rate of change of the angular momentum about the line of contact is not equal to the moment of the forces about that line.

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