6·3. Sudden changes and Initial motions. In **Pt. I, 15·41** we solved some problems on the effects of prescribed impulsive changes in the motion of a body, and in **Pt. I, 16·3** attention was called to the simplification of the equations of motion when a body or system starts from rest and only the initial values of the accelerations and stresses are required. As a further example of the first kind let us consider the change of motion produced in a lamina with a curved boundary by its impact on a fixed rough inelastic curve in its plane.

Scanned with CamScanner

Referring to the figure of 6·2, let Q be the point of impact, ω, ω' the angular velocities of the lamina just before and after the impact, and v the component velocity of G at right angles to GQ just before impact. Then since the moment of momentum about Q is unaltered by the blow, the equation

$$m(k^2+r^2)\omega' = mk^2\omega + mvr$$

determines the angular velocity with which the lamina begins to roll.

6·31. Initial curvature. *To determine the initial curvatures of the paths of assigned points of a system, when the system is set in motion in a given manner, or when it begins to move from rest owing to the giving way of a constraint.*

In the former case, if the velocity of an assigned point is known in magnitude and direction, since the normal acceleration v^2/ρ can be found from the equation of motion, therefore the curvature $1/\rho$ is determinate.

In the latter case, if we can find the initial direction of motion, and also find the small displacements which take place in a short interval of time along and perpendicular to the direction of motion, then we may be able to find the curvature by using Newton's formula

$$\frac{1}{\rho} = \lim \frac{2y}{x^2}.$$

Otherwise it may be necessary to take such an expression as

$$\frac{1}{\rho} = \pm \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which gives the curvature of the path of the point (x, y) at time t, and find the limit of this expression as $t \to 0$; or, if polar co-ordinates (r, θ) are used, the limit of

$$\frac{1}{\rho} = \pm \frac{r^2\dot{\theta}^3 + 2\dot{r}^2\dot{\theta} - r\ddot{r}\dot{\theta} + r\dot{r}\ddot{\theta}}{(\dot{r}^2+r^2\dot{\theta}^2)^{\frac{3}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

by dr pk singh

Scanned with CamScanner

Consider (1), and denote by $\dot{x}_0, \ldots$ the value of $\dot{x}, \ldots$ at time $t=0$. We have $\dot{x}_0 = \dot{y}_0 = 0$. Then, by Maclaurin's Theorem,

$$\dot{x}^2 + \dot{y}^2 = t^2 (\ddot{x}_0^2 + \ddot{y}_0^2) + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = \frac{t^2}{2} (\ddot{x}_0 \dddot{y}_0 - \dddot{x}_0 \ddot{y}_0) + \frac{t^3}{3} (\ddot{x}_0 \ddddot{y}_0 - \ddddot{x}_0 \ddot{y}_0) + \ldots \quad \ldots (4).$$

The values of x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ are known. The values of $\ddot{x}_0$, $\ddot{y}_0$ are obtained from the equations of motion; and the initial values of the higher derivatives are obtained by successive differentiations of the equations of motion. Usually, the coefficient of t^2 in (3) is positive, while the coefficient of t^2 in (4) is zero, and the coefficient of t^3 is not zero. Then the initial curvature is

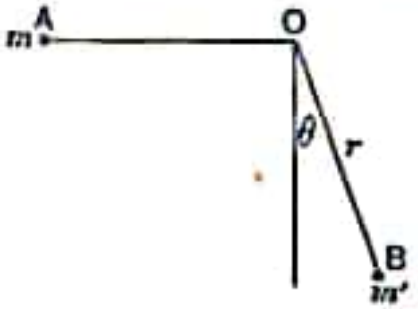
$$\pm \frac{\ddot{x}_0 \dddot{y}_0 - \dddot{x}_0 \ddot{y}_0}{3 (\ddot{x}_0^2 + \ddot{y}_0^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Similarly, in polar co-ordinates, the initial curvature is

$$\pm \frac{3\dot{r}_0^2 \ddot{\theta}_0^2 + 6\ddot{r}_0 \dot{r}_0 \dot{\theta}_0 + r_0 \dddot{r}_0 \dot{\theta}_0 - r_0 \dddot{r}_0 \ddot{\theta}_0}{3 (r_0^2 \ddot{\theta}_0^2 + \ddot{r}_0^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots (6).$$

6·32. Examples. (i) *Two particles A and B are connected by a fine string; A rests in limiting equilibrium on a rough horizontal table and B hangs vertically at a distance a below the edge of the table. The string being entirely in a plane at right angles to the edge, B is projected horizontally in this plane with velocity V. Prove that the initial acceleration of A is $\mu V^2/(\mu+1)a$, and that the initial curvature of the path of B is $1/a(\mu+1)$.*

Let m, m' denote the masses of A, B. Let the position of B be defined by polar co-ordinates r, θ referred to an origin at the edge of the table and such that initially $r=a$, $\theta=0$ and $r\dot{\theta}=V$.



For limiting equilibrium we must have $m'g = \mu mg$.

If T is the tension, then, in the initial position, the equations of motion are

$$m\ddot{r} = T - \mu mg$$

and $$m'(\ddot{r} - r\dot{\theta}^2) = m'g - T;$$

whence by addition $$(m+m')\ddot{r} = m'r\dot{\theta}^2 = m'V^2/a$$

or $$\ddot{r} = \mu V^2/(\mu+1)a.$$

Again, if ρ be the initial radius of curvature of the path of B,

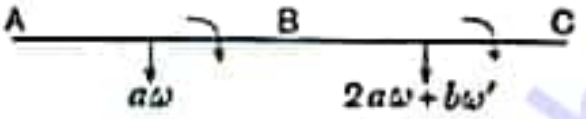
$$\frac{V^2}{\rho} = -(\ddot{r} - r\dot{\theta}^2) = -\frac{\mu V^2}{(\mu+1)a} + \frac{V^2}{a} = \frac{V^2}{(\mu+1)a},$$

so that $$\rho = (\mu+1)a.$$

Scanned with CamScanner

(ii) *Two uniform rods AB, BC are smoothly jointed at B and can turn about the end A which is fixed. Initially they are held so that ABC is a horizontal straight line and then they are let go. Find the initial curvature of the path of C.*

Let m, m' be the masses and $2a$, $2b$ the lengths of AB, BC, and ω, ω' their initial angular accelerations.



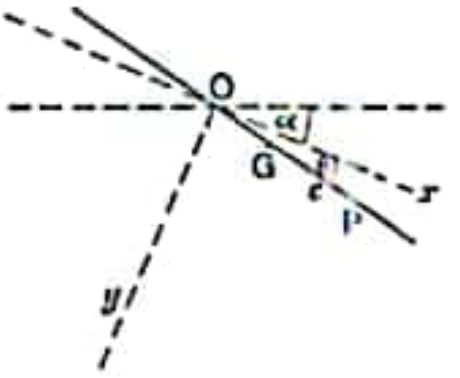
By taking moments about B for BC and about A for the system, we get
$$\tfrac{1}{3}m'b^2\omega' + m'b(2a\omega + b\omega') = m'gb$$
and
$$\tfrac{1}{3}ma^2\omega + \tfrac{1}{3}m'b^2\omega' + m'(2a\omega + b\omega')(2a+b) = mga + m'g(2a+b),$$
leading to
$$\frac{\omega}{\omega'} = -\frac{b}{a}\frac{m'+2m}{m}.$$

Suppose that, after a short time t, θ, ϕ are the inclinations of AB, BC to the horizontal; then the horizontal and vertical displacements of C are $2a(1-\cos\theta)+2b(1-\cos\phi)$ and $2a\sin\theta+2b\sin\phi$,

or, approximately, $a\theta^2+b\phi^2$ and $2a\theta+2b\phi$.

The initial tangent to the path of C is vertical, so that, if ρ is the radius of curvature,
$$2\rho = \lim\frac{4(a\theta+b\phi)^2}{a\theta^2+b\phi^2}.$$
But the first approximations to θ and ϕ are
$$\theta = \tfrac{1}{2}\omega t^2, \quad \phi = \tfrac{1}{2}\omega' t^2,$$
and hence we obtain
$$\rho = \frac{2ab(m+m')^2}{b(m+2m')^2+am^2}.$$

(iii) *A uniform thin rod of length $2a$ is held at an inclination α to the horizontal in contact with a smooth peg at its middle point. Prove that when the rod is let go the initial radius of curvature of the path of a point at a distance c from the middle point is $(a^2/c)\tan\alpha$.*



Let r, θ be the co-ordinates of P, so that $r-c$, θ are the co-ordinates of G. Then the equations of motion are
$$\ddot{r} - (r-c)\dot{\theta}^2 = g\sin(\alpha+\theta) \quad \ldots(1),$$
$$(r-c)[(r-c)\ddot{\theta} + 2\dot{r}\dot{\theta}] + \tfrac{1}{3}a^2\ddot{\theta} = g(r-c)\cos(\alpha+\theta) \quad \ldots(2).$$

Then $r_0 = c$, $\dot{r}_0 = 0$, $\theta_0 = 0$, $\dot{\theta}_0 = 0$. By (1), (2), $\ddot{r}_0 = g\sin\alpha$, $\ddot{\theta}_0 = 0$. We wish to differentiate (1) and (2), twice. We can therefore neglect small quantities of the 3rd order in powers of t. Since $\dot{r}_0 = 0$, $r-c$ is of the second

Scanned with CamScanner

order; and similarly, θ is of the 3rd order. Hence to the required degree of approximation, (1) and (2) become

$$\ddot{r} = g \sin\alpha, \quad a^2\ddot{\theta} = 3g(r-c)\cos\alpha,$$

giving $\dot{r}_0 = \dot{r}_0 = 0, \ \theta_0 = 0, \ \dot{\theta}_0 = 3g^2 \sin\alpha\cos\alpha/a^2.$

Hence, by 6·31 (6), the initial radius of curvature is $a^2 \tan\alpha/c$.

by dr. PN. Singh

Scanned with CamScanner