1.31. ORDER RELATION IN CUTS

Let $\alpha = (X_1, X_2)$ and $\beta = (Y_1, Y_2)$ be two cuts.

Then $\alpha < \beta$ if every member of X_1 is a member of Y_1 but every member of Y_1 is not a member of X_1, i.e. if X_1 is a proper subset of Y_1 i.e. $X_1 \subset Y_1$ (or what is the same if Y_2 is a proper subset of X_2.)

Equivalently, we say that $\beta > \alpha$.

Ex. ***For any positive cuts*** $\alpha = (X_1, X_2)$, $\beta = (Y_1, Y_2)$, ***prove that*** $(X_1, X_2) + (Y_1, Y_2)$ ***is positive i.e.*** $\alpha > 0, \beta > 0 \Rightarrow \alpha + \beta > 0$.

[M.U. 74H]

Proof : Let $\alpha = (X_1, X_2)$ and $\beta = (Y_1, Y_2)$.

Also, let $\gamma = (X_1, X_2) + (Y_1, Y_2) = (Z_1, Z_2)$.

Since α is positive, therefore $O_1 \subset X_1$.

Similarly $O_1 \subset Y_1$.

Since $z_1 \in Z_1$ is obtained by adding elements of X_1 with those of Y_1, we get $O_1 \subset Z_1$ which implies that (Z_1, Z_2) is a positive cut.

Thus $\alpha > 0$ (a positive cut), $\beta > 0 \Rightarrow \alpha + \beta > 0$.

(i) Trichotomy Law of Cuts

From the definition of cuts, it is evident that one and only one of the following relations holds.

(i) X_1 is a proper subset of Y_1

(ii) Y_1 is a proper subset of X_1

(iii) $X_1 = Y_1$.

From this, we conclude that for any two cuts

$$\alpha = (X_1, X_1),\ \beta = (Y_1, Y_2)$$

we have either

(i) $(X_1, X_2) < (Y_1, Y_2)$ or $(Y_1, Y_2) < (X_1, X_2)$ or $(X_1, X_2) = (Y_1, Y_2)$

i.e. either $\alpha < \beta$ or $\beta < \alpha$ or $\alpha = \beta$.

Thus the trichotomy law for the order relation in the system of cuts (real numbers) holds.

(ii) Transitivity of the order relation

Let $\alpha = (X_1, X_2)$, $\beta = (Y_1, Y_2)$ *and* $\gamma = (Z_1, Z_2)$ *be three cuts such that* $\alpha < \beta$ *and* $\beta < \gamma$ *i.e.* $(X_1, X_2) < (Y_1, Y_2)$ *and* $(Y_1, Y_2) < (Z_1, Z_2)$. *Then* $(X_1, X_2) < (Z_1, Z_2)$ *i.e.* $\alpha < \gamma$.

By definition, $X_1 \subset Y_1$ and $Y_1 \subset Z_1$.

This $\Rightarrow$ that $X_1 \subset Z_1$ and hence $(X_1, X_2) < (Z_1, Z_2)$ i.e. $\alpha < \gamma$.

Thus we have shown that $\alpha < \beta$ and $\beta < \gamma \Rightarrow \alpha < \gamma$.

1.32. TWO THEOREMS

Theorem I : *Let* $a = (A_1, A_2)$, $b = (B_1, B_2)$ *and* $c = (C_1, C_2)$ *be cuts of rational numbers. Then*

$$(A_1, A_2) > (B_1, B_2) \Rightarrow (A_1, A_2) + (C_1, C_2) > (B_1, B_2) + (C_1, C_2)$$

i.e. $$a > b \Rightarrow a + c > b + c.$$ **[Bhag. 74H]**

Proof : Let $a = (A_1, A_2)$, $b = (B_1, B_2)$ and $c = (C_1, C_2)$.

Let $(A_1, A_2) + (C_1, C_2) = (U_1, U_2)$

and $(B_1, B_2) + (C_1, C_2) = (V_1, V_2)$.

In order to prove that $(U_1, U_2) > (V_1, V_2)$, we need to prove that $V_1 \subset U_1$.

Now, from the definition,

$$U_1 = \{x_1 + z_1 \in Q \mid x_1 \in A_1, z_1 \in C_1\} \quad ...(1)$$

$$V_1 = \{y_1 + z_1 \in Q \mid y_1 \in B_1, z_1 \in C_1\} \quad ...(2)$$

It is given that $(A_1, A_2) > (B_1, B_2')$ i.e. $B_1 \subset A_1$ i.e. B_1 is a proper subset of A_1. Hence (2) in conjunction with (1) implies there are more rational points in U_1 than in V_1. That is, $V_1 \subset U_1$.

Hence the result follows :

Cor. $a \geq b \Rightarrow a + c \geq b + c$.

Theorem II : *Let* $a = (A_1, A_2)$, $b = (B_1, B_2)$, $c = (C_1, C_2)$ *be cuts of rational numbers. Then*

$$(A_1, A_2) < (B_1, B_2), (C_1, C_2) > (O_1, O_2)$$

$$\Rightarrow (A_1, A_2)(C_1, C_2) < (B_1, B_2)(C_1, C_2).$$

i.e. $a < b,\ c > 0 \Rightarrow ac < bc$ [P.U. 71H]

Proof : First of all we suppose that $a = (A_1, A_2)$ and $b = (B_1, B_2)$ both are positive.

Let $(A_1, A_2)(C_1, C_2) = (U_1, U_2)$ ...(1)

and $(B_1, B_2)(C_1, C_2) = (V_1, V_2)$ ...(2)

We need to prove that $U_1 \subset V_1$.

We observe first that since $a > 0$, $b > 0$, $c > 0$, therefore $ac > 0$, $bc > 0$

i.e. $(U_1, U_1) > (O_1, O_2)$ and $(V_1, V_2) > (O_1, O_2)$.

It is given that $(A_1, A_2) < (B_1, B_2)$ so that $A_1 \subset B_1$. It means that there are more positive rationals in B_1 than in A_1 and consequently there will be more positive rationals in V_1 than in U_1 and hence $U_1 < V_1$.

The other cases may be dealt with similarly.