. THEOREM

Between any two different real numbers, there are infinity of irrational numbers.

[B.U. 68H; Bhag. 67H, 90H; R.U. 70H; M.U. 68H, 73H]

Proof : Suppose $\alpha < \beta$.

We have proved in the preceding theorem that there lie an infinite number of rational numbers between α and β.

We pick up two rational numbers r_1 and r_2 from this infinite set of rationals such that

$$\alpha < r_1 < r_2 < \beta.$$

We want to prove that there exists an irrational number lying between r_1 and r_2.

For this, we consider the real number

$$r = r_1 + \frac{r_2 - r_1}{\sqrt{2}}.$$

Clearly r is an irrational real number and we find that

$$r > r_1 \text{ because } r_2 > r_1;$$

and also $r < r_2$ because

$$r_2 - r = r_2 - r_1 - \frac{r_2 - r_1}{\sqrt{2}}$$

$$= (r_2 - r_1)\left(1 - \frac{1}{\sqrt{2}}\right) \text{ which is } > 0.$$

Thus $\quad r_1 < r < r_2.$

Hence $\quad \alpha < r_1 < r < r_2 < \beta.$

Thus we have shown that there exists one irrational number r between α and β. But r_1 and r_2 are arbitrary rationals lying between α and β. Now since there are infinity of rational numbers r_1 and r_2 between α and β, hence we get infinity of irrational real numbers according to the scheme above. This proves the theorem.

.36. SECTION OF THE REAL NUMBERS : COMPLETENESS THEOREM

Let us recall that in art 1.13 we considered the sections of rational numbers i.e. we divided the set Q of rational numbers into two classes L and R characterised by the following properties :

(i) $L \neq \phi$, $R \neq \phi$ i.e., each class is non-empty.

(ii) Every rational number belongs to either L or R i.e. no rational number escapes classification.

(iii) Every member of L is less than every member of R i.e. $x \in L, y \in R \Rightarrow x < y$.

The two classes L and R are called the lower class and upper class respectively.

This type of division of the set Q into two classes is called a 'section'.

The set of all upper bounds of L is the set R and the set of all lower bounds of R is the set L. Sometimes, we use the term 'maximum' or 'greatest' for the least upper bound (lub) of a set when it belongs to the set. When the glb of a set is in the set, it is sometimes called the 'minimum' or 'least' of the set.

We have observed that this section of the set of rational numbers admits of four possibilities :

(i) L has a greatest member and R has a least member.

(ii) L has a greatest member and R has no least member.

(iii) L has no greatest member and R has a least member.

(iv) L has no greatest member and R has no least member.

The first possibility (i) is inadmissible.

We know that the possibility (ii) or possibility (iii) corresponds to the rational number. Thus the possibility (ii) or the possibility (iii) does not produce any new number different from what is at our hand, namely rational number.

The fourth possibility namely L has no greatest member and R has no least member gives rise to a new number (irrational number) which is different from the member of the underlying set Q.

This was amply demonstrated by the following classification of L and R :

$$L = \{x \in Q \mid x \text{ is } -ve, 0 \text{ or all } +ve\ x \text{ such that } x < \sqrt{2}\}$$

$$R = \{y \in Q \mid y > \sqrt{2}\}$$

As in the ex. above, (L, R) corresponds to a number $\sqrt{2}$ which is not a rational number.

The rational numbers and irrational numbers taken together are called real numbers.

Now, a natural question that arises in this connection is whether by cutting the set of real numbers into two classes L and R as we have done in the case of set of rational numbers shall we discover a new number different from a real number.

We have seen before that a new number (real number) different from the member of an underlying set Q arises when L has no greatest member and R has no least member. Hence the answer of the above question can be available by the following statement : If we divide the set of real numbers into two classes L and R as we have classified the set of rational numbers, then we shall expect to obtain a new number different from a real number only when L has no greatest real number and R has no least member, otherwise not.. The purpose of this article is to probe into these questions. We now proceed to classify the set of real numbers exactly on the same pattern as we have done in the case of rational numbers.